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## LETTER TO THE EDITOR

# The length of attractors in asymmetric random neural networks with deterministic dynamics

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**Abstract.** I have developed a method to detect attractors of any length in large neural networks with up to 1024 neurons within a reasonable period of CPU time. In networks with symmetric couplings only stable states and, in the case of parallel dynamics, cycles of length 2 exist. The presented simulations suggest that, in sufficiently large systems, this holds also for couplings up to a distinct value of asymmetry. Beyond this value extremely long cycles are detected and the average cycle length depends exponentially on system size.

In recent years attractor neural networks [1] have become an interesting field for many disciplines. Influenced mainly by biology, former network models [2, 3] have been modified, using other neural representations, e.g. McCulloch-Pitts instead of Ising neurons [4], or introducing asymmetric couplings [5-7].

The attractors of finite networks of two state-neurons with non-symmetric couplings are limit cycles as well as fixed points. These cycles can be formed by a suitable learning algorithm to store temporal sequences [8]. They can also be utilized to generate signals, oscillating with large time constants, corresponding to the respective cycle length [9].

The influence of coupling symmetry [7] and other system parameters [10] on number and properties of the *fixed points* have been calculated analytically. Concerning dynamics, analytical calculations exist for totally asymmetric and weak correlated couplings [11]. Numerical simulations, dealing with spread of damage and remanent magnetization, suggest a sharp transition between a *chaotic* phase at low and a *frozen* phase at high coupling symmetries [12].

In this letter cycles in networks with random Gaussian couplings are examined. The results of numerical simulations suggest a distinct critical value right in the middle between symmetry and asymmetry (see below), separating areas with and without extremely long cycles. Additional simulations, recently carried out, give rise to the assumption that this also holds for asymmetrically diluted Hebbian couplings and show something about the nature of these cycles.

The systems mainly considered are fully connected Gaussian networks of  $N$  Ising or McCulloch-Pitts neurons, respectively. Following the lines of Gutfreund *et al* [7], the couplings  $J_{ij}$  are obtained from

$$J_{ij} = J_{ij}^s + kJ_{ij}^a \quad (1)$$

where  $J_{ij}^s = J_{ji}^s$  is the symmetric and  $J_{ij}^a = -J_{ji}^a$  the antisymmetric component of the coupling matrix. The parameter  $k$  defines the *coupling symmetry* ranging from  $k = 0$  for symmetric through  $k = 1$  for asymmetric to  $k \rightarrow \infty$  for antisymmetric couplings. It is related to the more commonly used parameter  $\eta$  by

$$\eta = \frac{1 - k^2}{1 + k^2} = N \langle J_{ij} J_{ji} \rangle \quad (2)$$

ranging from  $\eta = 1$  for symmetric through  $\eta = 0$  for asymmetric to  $\eta = -1$  for antisymmetric couplings.

Here, as usual, the elements of the upper triangular matrices of the components  $J_{ij}^s$  and  $J_{ij}^a$  are random independent Gaussian variables with zero mean and mean square equal to  $1/N(1 + k^2)$ , which ensures that the rows of the coupling matrices have average norm 1, independent of system size and symmetry.

Two deterministic types of updating are considered, namely:

$$\text{parallel} \quad s_i(t+1) = \text{sgn} \left( \sum_{j \neq i} J_{ij} s_j(t) \right) \quad (3)$$

$$\text{sequential} \quad s_i(t+1) = \text{sgn} \left( \sum_{j < i} J_{ij} s_j(t+1) + \sum_{j > i} J_{ij} s_j(t) \right) \quad (4)$$

for Ising neurons ( $s_i = \pm 1$ ). For McCulloch–Pitts neurons ( $s_i = 0, 1$ ) the signum function has to be replaced by the step function

$$\Theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases} \quad (5)$$

Under these dynamical processes every initial configuration evolves to a definite attractor, which is in general a periodic repetition of  $l$  configurations, i.e. a cycle of length  $l$  with  $l \leq 2^N$ , the number of possible states. Thus, the configuration space is partitioned into separate basins of attraction.

For a given symmetry  $\eta$  a large number of samples has been randomly generated according to equation (1) with the corresponding value for  $k$ . For every sample with  $N \leq 16$  I have calculated the number  $n$  of attractors and the basins of attraction  $\Omega_s$  ( $s = 1, \dots, n$ ), where  $\Omega_s$  is the number of initial configurations, which flow to the  $s$ th attractor. Additionally, the average attractor size

$$\langle l \rangle = \left\langle \frac{1}{n} \sum_{s=1}^n l_s \right\rangle \quad (6)$$

and the basin-size-weighted average attractor size

$$\langle l' \rangle = \left\langle \sum_{s=1}^n \frac{\Omega_s}{2^N} l_s \right\rangle \quad (7)$$

are averaged over all samples.

For  $N > 16$  for every sample one random initial configuration with activity 0.5 is generated. After a number of updates, every system reaches an attractor. Its length

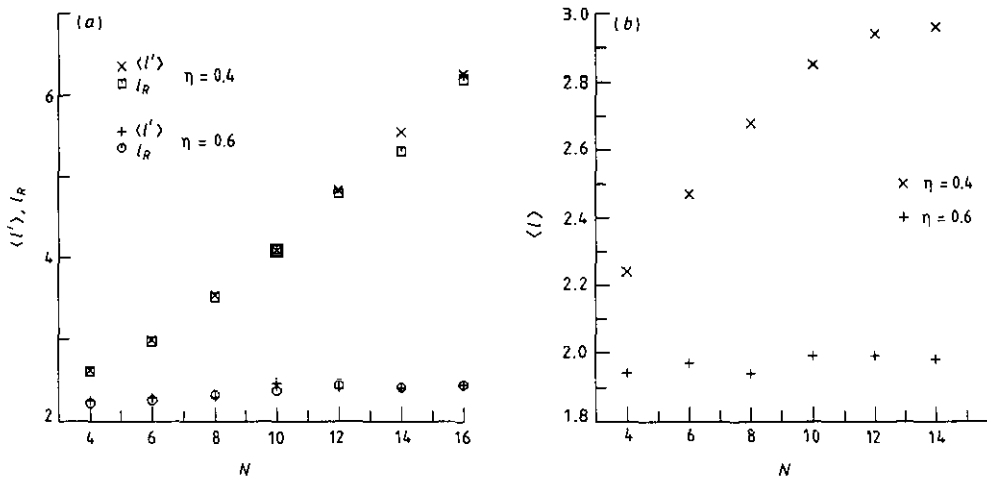


Figure 1. (a) Basin size weighted average attractor size  $\langle l' \rangle$  and  $l_R$  and (b) unweighted average attractor size  $\langle l \rangle$  against the system size  $N$  for two different coupling symmetries  $\eta$  with Ising neurons and parallel dynamics.

$l$  and the number of updates  $t$  are registered. By averaging over all samples we get values, denoted as  $l_R$  and  $t_R$ . Note that  $l_R$  should take values, comparable to those of  $\langle l' \rangle$ .

Figure 1(a) shows the basin-size-weighted average attractor size  $\langle l' \rangle$  and  $l_R$  in small Ising neuron systems for coupling symmetries  $\eta = 0.6$  and  $\eta = 0.4$  calculated with parallel dynamics. The  $\langle l' \rangle$  are average values of 2500, the  $l_R$  of 10 000 samples. As expected, nearly equivalent values for  $\langle l' \rangle$  and  $l_R$  are obtained.

In contrast to the results for  $\eta = 0.6$ , the values of  $\langle l' \rangle$  and  $l_R$  for  $\eta = 0.4$  clearly increase with system size. This also holds for the corresponding unweighted average attractor sizes  $\langle l \rangle$  (figure 1(b)), but these values are smaller, which means that longer attractors have larger basins of attraction.

In figure 2 the logarithm of  $l_R$  is plotted against the system size  $N$  for networks with between 16 and 80 Ising neurons and six different coupling symmetries. For  $\eta > 0.5$  there is no increase of the average cycle length with system size. Rather, for very large systems (i.e.  $N \rightarrow 1024$ ) the values seem to converge slowly to  $\ln 2$ . In fact, with  $\eta > 0.5$  and for sufficiently large systems, for parallel dynamics nearly exclusively cycles of length 2 are detected. Their basins of attraction dominate the configuration space, so that no stable states are found, although they exist, as simulations with sequential dynamics show (note that stable states are fixed points under both kinds of dynamics).

For  $\eta \simeq 0.5$  no results are plotted, because the values, obtained by averaging over one set of 1000 samples, often differed strongly from those from another set of 1000 samples.

For  $\eta < 0.5$ , however, there is an exponential dependence  $l_R \propto \exp(aN)$  with values  $a$  of 0.05, 0.08 and 0.12 in the case of  $\eta = 0.45$ , 0.4 and 0.34, respectively, obtained by a least square fit to the simulation data in figure 2. Note that the  $l_R$  do not give typical cycle lengths, but are dominated by few very long cycles. If one would only plot the longest detected cycle length for given  $\eta$  and  $N$  against the system size

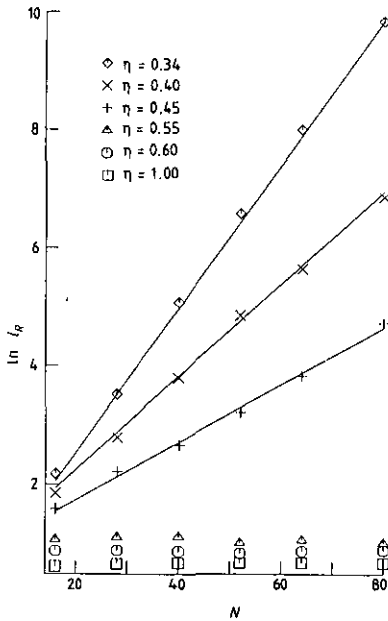


Figure 2. Logarithm of  $l_R$  against the system size  $N$  for six different coupling symmetries  $\eta$ .

$N$ , one would roughly get also an exponential increase. For  $\eta = 0.34$ ,  $N = 80$  e.g., the maximum length of the detected cycles was 493 298.

An analogous behaviour is observed at  $\eta = -0.5$ . For  $\eta < -0.5$  again  $l_R$  does not increase with system size, but the values seem to converge to  $\ln 4$ . For  $\eta < -0.5$  and sufficiently large systems, for parallel dynamics nearly exclusive cycles of length 4 are detected. The minima of the average number of attractors  $n$  and the maxima of  $\langle l \rangle$ ,  $\langle l' \rangle$  and  $l_R$  are at  $\eta = 0$  (see figures 11 and 12 of [7]).

With this method it was not possible to analyse much larger systems. The reason is that all systems should be relaxed into an attractor, and the number of necessary updates increases drastically with system size and as  $\eta \rightarrow 0$ . Nevertheless, for couplings with high symmetries, systems up to  $N = 1024$  could be simulated. In figure 3 the logarithm of  $t_R$ , the average number of updates, which were necessary to relax the system into an attractor, is plotted against the logarithm of the system size  $N$ . The full lines are the results of least-square fits to the simulation data for  $N \geq 192$ . A power law  $t_R \propto N^b$  is observed with values  $b$  of 0.63, 0.64 and 0.79 for  $\eta = 1$ , 0.98 and 0.88, respectively.

In contrast to the case of  $l_R$ , where the behaviour changes drastically at  $\eta \simeq 0.5$  the  $N$ -dependence of  $t_R$  changes smoothly with  $\eta$ . The power-law behaviour at  $\eta = 1$  gradually changes into an exponential dependence for  $0.5 > \eta > -0.5$  and back to a power law at  $\eta = -1$ . E.g. for  $\eta = 0.8$ , the increase of  $t_R$  with  $N$  is slower than

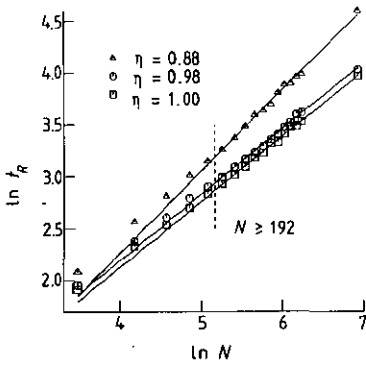


Figure 3. Logarithm of the average number of updates  $t_R$ , necessary to relax a system of size  $N$  from a random initial configuration to an attractor for high values of symmetry  $\eta$ .

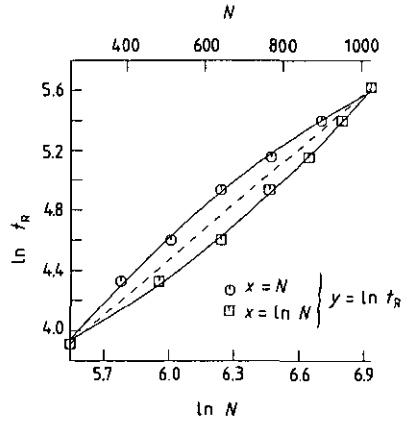


Figure 4. Logarithm of  $t_R$  against  $N$  (circles) and  $\ln N$  (squares) for  $\eta = 0.8$ . The increase is slower than exponential, but steeper than any power law.

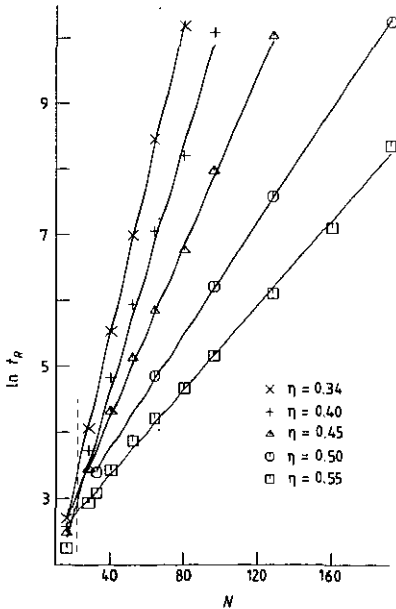


Figure 5. Logarithm of  $t_R$  against the system size  $N$  for coupling symmetries near  $\eta = 0.5$ .

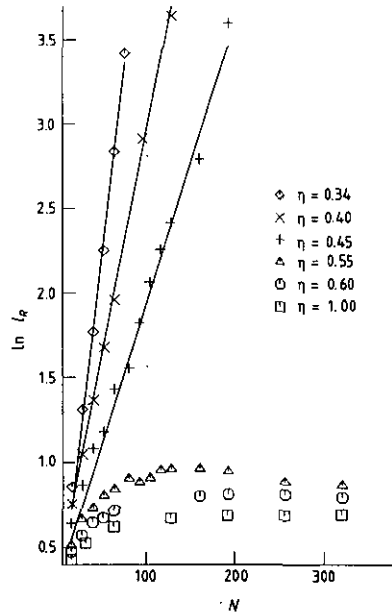


Figure 6. The same parameters as in figure 2, but for McCulloch-Pitts neurons.

exponential, but steeper than any power (see figure 4).

The exponential dependence  $t_R \propto \exp(cN)$  is shown in figure 5. The full lines were obtained by a least square fit to the simulation data for  $N \geq 28$ . The values for  $c$  range from 0.03 for  $\eta = 0.55$  to 0.12 for  $\eta = 0.34$ . The average number of necessary

updates, e.g. for  $N = 80$  and  $\eta = 0.34$ , is 26 520. While  $l_R$  does not give typical values, as mentioned above,  $t_R$  does.

A qualitatively equivalent behaviour is observed for sequential dynamics. For  $\eta > 0.5$  in large systems nearly exclusively *stable states*, for  $\eta < -0.5$  cycles of length 2 are detected. Again, for high coupling symmetries, the number of necessary updates obeys a power law  $t_R \propto N^b$ . At  $\eta = 1$  one finds  $b = 0.5$ , i.e.  $t_R \propto \sqrt{N}$ .

Likewise for McCulloch–Pitts neurons with coupling symmetries  $\eta > 0.5$ , the average cycle length  $l_R$  does not increase with system size for sufficiently large systems (figure 6). For  $\eta < 0.5$  the slopes of the full lines, obtained by least-square fits to simulation data with  $N \geq 28$ , are 0.017, 0.026 and 0.044 for  $\eta = 0.45, 0.4$  and  $0.34$ , respectively. The values are clearly smaller, than for Ising neurons.

For high coupling symmetries the average number of necessary updates  $t_R$  obeys a power law,  $t_R \propto N^b$ , with  $b = 0.5$  for parallel and  $b = 0.38$  for sequential dynamics at  $\eta = 1$ . For  $\eta < 0.5$   $t_R$  increases exponentially with system size,  $t_R \propto \exp(cN)$  with  $c = 0.025$  for  $\eta = 0.45$  and  $c = 0.043$  for  $\eta = 0.34$  for parallel dynamics.

While for Ising neurons the longest cycles and the lowest numbers of attractors are detected at  $\eta = 0$ , for McCulloch–Pitts neurons the maxima of  $\langle l \rangle$  and the minima of  $n$  are at  $\eta \simeq -\frac{1}{3}$ . The longest cycles, i.e. the maxima of  $\langle l \rangle$  and  $l_R$ , are found at a slightly different value  $\eta \simeq -0.5$ . Moreover, the average cycle length  $l_R$  at  $\eta = -1$  still depends exponentially on system size,  $l_R \propto \exp(aN)$  with  $a \simeq 0.05$  for parallel dynamics.

In conclusion the analysis of the complete configuration space of *small* systems (figure 1) shows no increase of the average cycle length as long as  $|\eta| > 0.5$  for Ising neurons and  $\eta > 0.5$  for McCulloch–Pitts neurons, respectively. Furthermore, longer cycles have larger basins of attraction.

If this observations also hold for larger systems, then the presented simulation data suggest that in sufficiently large systems with coupling symmetries  $\eta > 0.5$  no cycles with a length, larger than 1 for sequential and 2 for parallel dynamics, exist.

The critical value of  $\eta$  is independent of the representation of the neurons. The two cases considered (Ising and McCulloch–Pitts neurons) are extreme. The behaviour changes gradually if the value  $\alpha$ , taken for the resting neurons is changed gradually from  $-1$  to  $0$ , while systems with  $\alpha < -1$  behave like systems with  $1/\alpha$ .

In order to examine the nature of the cycles, additionally simulations with asymmetrically diluted *Hebbian* couplings have been (and will be) carried out. The results will be published in detail elsewhere, but some points of interest should be mentioned here.

The symmetry of Hebbian matrices from  $p = \alpha N$  random patterns can gradually be changed to  $\eta \rightarrow 0$  by asymmetrical dilution. If the degree of dilution is sufficiently weak, i.e. if the symmetry parameter  $\eta$  is higher than a critical value  $\eta_c$ , the attractor, reached by relaxation of a retrieval state, has a high *overlap*,  $q \simeq 0.97$ , with this state.  $\eta_c$  depends on the loading parameter  $\alpha$  and ranges from a value close to 0 for  $\alpha \rightarrow 0$  to  $\eta_c \rightarrow 1$  for  $\alpha \rightarrow \alpha_c = 0.14$  (a similar problem has been investigated for Gaussian couplings in [12], mentioned at the beginning of this letter). But, concerning *cycle lengths* the value  $\eta = 0.5$  in this case plays the same role as in the rest of this letter independent of the loading parameter. Also, in the case of low memory loading with random initial states, cycles with lengths  $\gg 2$  are detected if  $\eta < 0.5$ . In this case however, the cycles observed are rather short compared with the case of Gaussian couplings (also see [13]). Moreover, the portion of spin flips per update is only a few percent and therefore the overlaps of the cycle states with the retrieval states hardly

fluctuate in time. These overlaps show that the cycles do not, e.g., consist of a series of retrieval states, but are rather 'flowing around' a *mixture state* of these retrieval states. While such mixture states exist at all values of  $\eta$ , the formation of cycles around them in sufficiently large systems seems to be restricted to couplings with  $\eta < 0.5$ . Flip rates and cycle lengths increase with  $\alpha$ , and for Gaussian couplings the average flip rate reaches 50%.

The unexpected existence of a critical symmetry value concerning cycle lengths seems not to be restricted to the random Gaussian couplings examined in detail in this letter. To find the range of its validity and especially the reason for it will be one subject of future studies.

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